A Connection Between Strangeness-Free Delay Differential-Algebraic and Neutral Type Systems ⋆

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Abstract: We present an approach that allows to reduce a system of delay differential-algebraic equations to neutral type system with a prescribed additional dynamics. The approach avoids a transformation that divides the system into differential-difference, difference and algebraic parts. We show how the result can be applied to the computation of $\mathcal{H}_2$ norm for the delay differential-algebraic system. We assume that the system is strangeness-free that is less conservative than the standard assumption on the delay-free part of the system, which does not take into account the delayed term of the system.

Keywords: delay differential-algebraic equations, H_2 norm, stability analysis, neutral type, singular systems.

1. INTRODUCTION

In this paper, we analyze linear time-invariant systems of delay differential-algebraic equations (DDAEs), which are also called singular differential-difference systems. One of the motivations for the reduction of such systems to the systems of neutral type differential-difference equations (NDDEs) is the fact that some control problems, like characterizing of $\mathcal{H}_2$ norm of transfer function for systems of DDAEs, and computing exponential estimates, have never been addressed in the literature to the best of our knowledge. Also, some problems such as stability and robust stability analysis (see, e. g., Han et al. (2004); Michiels and Vyhlídal (2005); Kharitonov (2013); Michiels and Niculescu (2014); Gomez et al. (2016)), construction of the exponential estimates for the solutions (see, e. g., Hale and Verduyn-Lunel (1993); Kharitonov (2005); Kharitonov et al. (2005); Chashnikov and Egorov (2015)), to name a few, are better investigated for the neutral type systems than for systems of DDAEs.

In contrast to differential-difference systems, for some classes of linear time-invariant systems of DDAEs one can not even guarantee some basic properties, like continuability and uniqueness of the solutions. The most convenient systems for analysis are the so-called coupled differential-difference systems; see, e. g., Răsvan (1995); Fridman (2002); Pepe et al. (2008); Gu and Liu (2009). In this paper, we consider a broader class – the class of strangeness-free systems, i. e., systems, which can be divided by a linear transformation into three parts: differential-difference, difference and algebraic. Such "triangular" form has been presented in Du et al. (2013), where the authors also show, how to reduce the system to the system of NDDEs by differentiation and time-shift operation. The differentiation obviously introduce an additional dynamics into the system, but, as has been shown in Du et al. (2013), such dynamics does not break the stability under some natural constraints on the initial states. The approach has been applied to prove the fundamental property that the exponential stability of systems of strangeness-free DDAEs is equivalent to the negativity of the spectral abscissa.

The idea of reduction of DDAEs to NDDEs by introducing some additional dynamics is developed in our contribution. The approach does not rely on the transformation of the system to triangular form, and leads to the neutral type systems with a prescribed additional dynamics that allows to analyze stability and solve some other related problems via known technics for neutral type differential-difference systems. In this paper, we concern only one application of the developed method – computation of $\mathcal{H}_2$ norm of transfer function for systems of DDAEs.

We focus on two classes of systems of DDAEs. The first class consists of systems, such that the delay-free part is strangeness-free. The second class is broader than the first one, as it consists of all strangeness-free systems, in the sense of Du et al. (2013). We extract the first class for convenience in reading, as all the formulas and proofs are simpler for such systems, allowing to gain greater insight into the main ideas of our research.

⋆ This work was supported by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office, by the Optimization in Engineering Center OPTEC of the KU Leuven, the Research Foundation-Flanders, and by the project UCoCoS, funded by the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant Agreement No 675080. Alexey Egorov acknowledge Saint-Petersburg State University for a research grant 9.42.1396.2015.

The organization of the paper is as follows. We introduce basic definitions in Section 2, some auxiliary results in Section 3. Section 4 is devoted to reducing the system of DDAEs to system of NDDEs. In Section 5 we show how to apply the result to the stability analysis and $\mathcal{H}_2$ norm characterizing for controllable systems. Two illustrative examples are given in Section 6. Some concluding remarks end the paper.

2. THE SYSTEM

Consider a linear system of the form

$$\frac{d}{dt}\big(Ex(t)\big) = A_0x(t) + A_1x(t-h), \tag{1}$$

where $E, A_0, A_1 \in \mathbf{R}^{n\times n},\ x(t) \in \mathbf{R}^n$.

We focus on the case of singular E. Moreover, all the results are also valid for $E = 0$, i. e., if the system is difference-algebraic.

Let

$$H(s) = sE - A_0 - e^{-sh}A_1$$

be the characteristic matrix for system (1), and

$$\Lambda = \big\{ s \in \mathbf{C} \ \big|\ \det H(s) = 0 \big\}$$

be the spectrum.

In this paper, we consider two classes of such systems. The second class is considerably broader than the first one, and the first class can be considered like an illustration of our ideas, as all the formulas are simpler.

The first class is described by the following assumption.

Assumption 1. The delay-free system

$$\frac{d}{dt}\big(Ey(t)\big) = A_0y(t) \tag{2}$$

is strangeness-free, i. e., there exists a nonsingular matrix

$$T_1 = \begin{pmatrix} R \\ P \end{pmatrix} \in \mathbf{R}^{n\times n},$$

where R and P are some blocks of rows, such that $PE = 0$ and the matrix

$$S_1 = \begin{pmatrix} RE \\ PA_0 \end{pmatrix}$$

is invertible.

Introduce an auxiliary result, which shows the equivalence between this condition and the rather standard assumptions on the system.

Lemma 2. The following statements are equivalent:

1. System (2) is strangeness-free.

2. There exist two nonsingular matrices $F_1,\ F_2 \in \mathbf{R}^{n\times n}$, such that

$$F_1EF_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad F_1A_0F_2 = \begin{pmatrix} A_0^{(1)} & A_0^{(2)} \\ 0 & I \end{pmatrix}. \tag{3}$$

3. The pair (E, A_0) is regular and impulse-free, i. e., there exist two nonsingular matrices $F_3,\ F_4 \in \mathbf{R}^{n\times n}$, such that

$$F_3EF_4 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad F_3A_0F_4 = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix},$$

where J is a Jordan block matrix.

The dimension of the first blocks of all these matrices is $r_1 \times r_1$, where $r_1 = \operatorname{rank}(E)$.

Here and later I and 0 are identity and null matrices of appropriate dimension.

Proof. First, both Item 2 and Item 3 imply Item 1, as for Assumption 1 we can take $T_1 = F_1$ or $T_1 = F_3$, respectively. If Item 1 holds true, we can take

$$F_1 = T_1, \quad F_2 = S_1^{-1},$$
$$F_3 = \begin{pmatrix} \Omega^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -A_0^{(2)} \\ 0 & I \end{pmatrix} T_1,$$
$$F_4 = S_1^{-1} \begin{pmatrix} \Omega & 0 \\ 0 & I \end{pmatrix}$$

to prove Items 2 and 3. Here Ω is such that $J = \Omega^{-1}A_0^{(1)}\Omega$ is a Jordan canonical form for the square matrix $A_0^{(1)}$, and

$$\begin{pmatrix} A_0^{(1)} & A_0^{(2)} \end{pmatrix} = RA_0S_1^{-1}.$$

Remark 3. By Item 2 of the lemma, the system of coupled difference-differential equations

$$\dot{x}(t) = Ax(t) + By(t-h),$$
$$y(t) = Cx(t) + Dy(t-h),$$

is a particular case of the class, described by Assumption 1. Indeed, the substitution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix}$$

reduces the matrices of the system to the form (3).

Item 3 of the lemma is a common assumption, connected to the Weierstraß canonical form for deferential-algebraic delay-free systems; see, e. g., Kunkel and Mehrmann (2006).

Define the second class of systems.

Assumption 4. System (1) is strangeness-free (see, Du et al. (2013)), i. e., there exists a nonsingular matrix

$$T_2 = \begin{pmatrix} R \\ P_1 \\ P_2 \end{pmatrix} \in \mathbf{R}^{n\times n},$$

such that $P_1E = 0$, $P_2E = 0$, $P_2A_0 = 0$, and the matrix

$$S_2 = \begin{pmatrix} RE \\ P_1A_0 \\ P_2A_1 \end{pmatrix}$$

is invertible.

Note that this assumption, in contrast to the previous one, takes into account the delayed term of the system. If system (10) is strangeness-free, premultiplying by T_2 leads to the "triangular" form

$$\begin{aligned} \frac{d}{dt}\big(REx(t)\big) &= RA_0x(t) + RA_1x(t-h), \\ 0 &= P_1A_0x(t) + P_1A_1x(t-h), \\ 0 &= P_2A_1x(t-h), \end{aligned} \tag{4}$$

which consists of three parts: the set of differential-difference equations, the set of difference equations, and the set of algebraic equations. As has been shown in Du et al. (2013), such system is exponentially stable if and only if the spectrum Λ locates in the open left half-plane and is separated from the imaginary axis.

Next we present a mathematically rigorous approach that allows to reduce a strangeness-free system of DDAEs to a system of NDDEs with a prescribed dynamics, but here we describe the basic idea. The approach is equivalent to the following four steps:

1. Premultiplying a strangeness-free system by T_2 to reduce it to the "triangular" form (4).

2. Applying a time-shift operation to the third part of (4).

3. Applying an operation which can be formally written as

$$I\frac{d}{dt} - M,$$

where M is a Hurwitz matrix, to the second and the third parts of (4).

4. Premultiplying the obtained system by T_2^{-1}.

In our algorithm all these steps are hidden. As will be shown in what follows, the approach does not even need the computation of T_2.

3. AUXILIARY RESULTS

To reduce DDAEs to NDDEs we need to introduce some auxiliary elements. Consider two classes of systems separately.

3.1 The System Under Assumption 1

As $r_1 = \mathrm{rank}(E) < n$, there exists a matrix $P \in \mathbf{R}^{(n-r_1)\times n}$ of full rank, such that

$$PE = 0.$$

This matrix is a basis of the left null space of matrix E. In fact, this is a part of matrix T_1 in Assumption 1. But we do not need to compute the first part R, as we are not going to make the transformation. Instead, we need the second element – matrix $X \in \mathbf{R}^{n\times(n-r_1)}$, such that

$$PX = M^{-1},$$

where M is an arbitrary fixed real invertible matrix of dimension $(n - r_1) \times (n - r_1)$. As will be clear from what follows, it is better to choose a Hurwitz matrix.

Now we can prove a necessary and sufficient condition that allows to check, weather the Assumption 1 holds or not, without transformation of the system.

Theorem 5. Assumption 1 holds true if and only if the number

$$r_2 = \mathrm{rank}(E + XPA_0)$$

is equal to n.

Proof. As $\mathrm{rank}(X) = n - r_1$, there exists a matrix $R \in \mathbf{R}^{r_1\times n}$, such that $\mathrm{rank}(R) = r_1$ and $RX = 0$. It is easy to see that the square matrix

$$T_1 = \begin{pmatrix} R \\ P \end{pmatrix}$$

is invertible. Indeed, otherwise there exists a non-zero vector $q = (q_1^T\ q_2^T)^T$, such that

$$q_1^T R + q_2^T P = 0.$$

By definition of R and P

$$\begin{pmatrix} R \\ P \end{pmatrix} X = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}.$$

Therefore, $q_2^T M^{-1} = 0$ that implies that $q_2 = 0$. Hence, $q_1^T \neq 0$ and $q_1^T R = 0$. This is impossible, as matrix R is of full rank.

Premultiply $E + XPA_0$ by T_1:

$$T_1(E + XPA_0) = \\ = \begin{pmatrix} RE \\ PE \end{pmatrix} + \begin{pmatrix} RXPA_0 \\ PXPA_0 \end{pmatrix} = \begin{pmatrix} RE \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}PA_0 \end{pmatrix}.$$

Rank of this matrix is equal to n if and only if system (2) is strangeness-free.

3.2 The System Under Assumption 4

Consider now more general case, which is also more complicated. Assume that we computed matrices P and X from the previous subsection, but $r_2 = \mathrm{rank}(E + XPA_0) < n$. This means that system (1) does not satisfy Assumption 1. We need to check now Assumption 4. Construct an additional element – a full rank matrix $\widetilde{P}_2 \in \mathbf{R}^{(n-r_2)\times(n-r_1)}$ such that

$$\widetilde{P}_2 PA_0 = 0,$$

i. e., whose rows are basis vectors of the left null space of matrix PA_0. Choose $\widetilde{P}_1 \in \mathbf{R}^{(r_2-r_1)\times(n-r_1)}$ such that

$$\begin{pmatrix} \widetilde{P}_1 \\ \widetilde{P}_2 \end{pmatrix}$$

is invertible. Now we can define $P_1 = \widetilde{P}_1 P$ and $P_2 = \widetilde{P}_2 P$.

Fix two invertible matrices $M_1 \in \mathbf{R}^{(r_2-r_1)\times(r_2-r_1)}$, $M_2 \in \mathbf{R}^{(n-r_2)\times(n-r_2)}$, and compute $X_1 \in \mathbf{R}^{n\times(r_2-r_1)}$, $X_2 \in \mathbf{R}^{n\times(n-r_2)}$, such that

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} (X_1\ X_2) = \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix}.$$

They also can be found as a solution of the system

$$(X_1\ X_2) \begin{pmatrix} M_1\widetilde{P}_1 \\ M_2\widetilde{P}_2 \end{pmatrix} = XM.$$

Theorem 6. Assumption 4 holds true, i. e., system (1) is strangeness-free, if and only if the number

$$r_3 = \mathrm{rank}(E + X_1P_1A_0 + X_2P_2A_1)$$

is equal to n.

Proof. There exists a full rank matrix $R \in \mathbf{R}^{r_1\times n}$, such that $R\,(X_1\ X_2) = 0$. The square matrix

$$T_2 = \begin{pmatrix} R \\ P_1 \\ P_2 \end{pmatrix}$$

is invertible.

Premultiply $E + X_1P_1A_0 + X_2P_2A_1$ by T_2:

$$T_2(E + X_1P_1A_0 + X_2P_2A_1) = \begin{pmatrix} RE \\ M_1^{-1}P_1A_0 \\ M_2^{-1}P_2A_1 \end{pmatrix}.$$

This matrix is nonsingular if and only if system (1) is strangeness-free.

Remark 7. An obvious corollary of the presented theorem is that the strangeness-free property is invariant with respect to the choice of matrices M_1, M_2, P_1, P_2, X_1, X_2.

4. REDUCING TO THE NEUTRAL TYPE DELAY SYSTEMS

Using the introduced elements, we can construct neutral type system with a prescribed dynamics determined by the chosen matrices M, M_1, M_2 for system (1).

4.1 The system under Assumption 1

Introduce the system

$$\frac{d}{dt}\Big((E + XPA_0)z(t) \ +XPA_1z(t-h)\Big) \\ = A_0z(t) + A_1z(t-h). \tag{5}$$

Theorem 8. The spectrum of system (5) is equal to

$$\Lambda \cup \sigma(M),$$

where $\sigma(M)$ is the spectrum of matrix M.

Proof. Let

$$H_1(s) = s(E + XPA_0) + sXPA_1e^{-sh} - A_0 - A_1e^{-sh}$$

be the characteristic matrix of system (5). It is easy to see that

$$\begin{aligned} H_1(s) &= H(s) + sXP(A_0 + A_1e^{-sh}) \\ &= H(s) + sXP(sE - H(s)) \\ &= (I - sXP)H(s) + s^2XPE \\ &= (I - sXP)H(s). \end{aligned} \tag{6}$$

Applying of Schur's formulas finishes the proof:

$$\begin{aligned} \det\ (I - sXP) &= \det(I - sPX) \\ &= \det(I - sM^{-1}) = \det(M^{-1})\det(M - sI). \end{aligned}$$

Remark 9. By Theorem 5, system (5) is a non-singular system of neutral type.

4.2 The System Under Assumption 4

Now we present two neutral type systems, corresponding to the nominal system (1) under Assumption 4. The form of the first system is simpler, whereas the structure of the spectrum is more complicated than for the second one.

I. The first neutral type system

Consider the system

$$\frac{d}{dt}\Big((E \ +X_1P_1A_0 + X_2P_2A_1)z(t) \\ +X_1P_1A_1z(t-h)\Big) = A_0z(t) + A_1z(t-h). \tag{7}$$

Theorem 10. The spectrum of system (7) is equal to

$$\Lambda \cup \sigma(M_1) \cup \Lambda_2,$$

where Λ_2 is the spectrum of the thoroughly studied retarded type system

$$\dot{y}(t) = M_2y(t-h). \tag{8}$$

Proof. The characteristic matrix of system (7)

$$\begin{aligned} H_2(s) = s(E \ &+X_1P_1A_0 + X_2P_2A_1) \\ &+se^{-sh}X_1P_1A_1 - A_0 - e^{-sh}A_1. \end{aligned}$$

By definition of $H(s)$,

$$\begin{aligned} H_2(s) = H(s) \ &+sX_1P_1(A_0 + A_1e^{-sh}) + sX_2P_2A_1 \\ = H(s) \ &+sX_1P_1(sE - H(s)) \\ &+se^{sh}X_2P_2(sE - A_0 - H(s)). \end{aligned}$$

Taking into account the definition of matrices P_1, P_2, we get

$$H_2(s) = (I - sX_1P_1 - se^{sh}X_2P_2)H(s).$$

As $P_2X_1 = 0$,

$$\begin{aligned} I- \ &sX_1P_1 - se^{sh}X_2P_2 \\ &= I - sX_1P_1 - se^{sh}X_2P_2 + s^2e^{sh}X_2P_2X_1P_1 \\ &= (I - se^{sh}X_2P_2)(I - sX_1P_1). \end{aligned}$$

Schur's formulas help to prove the desired result.

II. The second neutral type system

Introduce now another system

$$\begin{aligned} \frac{d}{dt}\Big((E \ &+X_1P_1A_0 + X_2P_2A_1)z(t) \\ &+ \ X_1P_1A_1z(t-h)\Big) = (A_0 + X_2M_2P_2A_1)z(t) \\ &+ \ (I - X_2M_2P_2)A_1z(t-h). \end{aligned} \tag{9}$$

Theorem 11. The spectrum of system (9) is equal to

$$\Lambda \cup \sigma(M_1) \cup \sigma(M_2).$$

Proof. The characteristic matrix of system (9)

$$\begin{aligned} H_3(s) \ &= s(E + X_1P_1A_0 + X_2P_2A_1) + sX_1P_1A_1e^{-sh} \\ &-(A_0 + X_2M_2P_2A_1) - (I - X_2M_2P_2)A_1e^{-sh}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} H_3(s) \ &= H(s) + sX_1P_1(A_0 + A_1e^{-sh}) \\ &+sX_2P_2A_1 - (1 - e^{-sh})X_2M_2P_2A_1 \\ &= H(s) + sX_1P_1(sE - H(s)) \\ &+se^{sh}X_2P_2(sE - A_0 - H(s)) \\ &-(e^{sh} - 1)X_2M_2P_2(sE - A_0 - H(s)). \end{aligned}$$

By the definition of P_1, P_2, we get

$$\begin{aligned} H_3(s) \ &= (I - sX_1P_1 - se^{sh}X_2P_2 \\ &+(e^{sh} - 1)X_2M_2P_2)H(s) = Q(s)H(s), \end{aligned}$$

where

$$Q(s) = \\ = \left[I - (X_1 \ \ X_2)\begin{pmatrix} sI & 0 \\ 0 & se^{sh}I - (e^{sh} - 1)M_2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right].$$

Compute the determinant of $Q(s)$, using Schur's formulas:

$$\begin{aligned} &\det\left(I - (X_1 \ \ X_2)\begin{pmatrix} sI & 0 \\ 0 & se^{sh}I - (e^{sh} - 1)M_2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right) \\ &= \det\left(I - \begin{pmatrix} sI & 0 \\ 0 & se^{sh}I - (e^{sh} - 1)M_2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}(X_1 \ \ X_2)\right) \\ &= \det\left(I - \begin{pmatrix} sI & 0 \\ 0 & se^{sh}I - (e^{sh} - 1)M_2 \end{pmatrix}\begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix}\right) \\ &= \det(I - sM_1^{-1})\det(e^{sh}I - se^{sh}M_2^{-1}). \end{aligned}$$

Thus, the result is proven.

Remark 12. By Theorem 6, systems (7) and (9) are non-singular systems of neutral type.

5. APPLICATION TO THE STABILITY ANALYSIS AND $\mathcal{H}_2$ NORM CHARACTERIZING

5.1 Stability Analysis

Assume that system (1) is strangeness-free. We need to choose matrices M_1, M_2 from Subsection 3.2. For simplicity, take $M_1 = \alpha_1 I$, $M_2 = \alpha_2 I$, where α_1, α_2 are some real constants. Obviously, we have to take (see, Andronov and Maier (1946)) $\alpha_1 < 0$, $\alpha_2 \in \left(-\frac{\pi}{2h}, 0\right)$ to guarantee that the additional spectrum of system (7) is located in the left half of the complex plane.

In this case systems (1) and (7) are equivalent in the sense of stability. Applying an existing method for the stability analysis of the neutral type system (7) (see, e. g., Han et al. (2004); Michiels and Vyhlídal (2005); Kharitonov (2013); Michiels and Niculescu (2014); Gomez et al. (2016)), we can define the stability of (1).

Also, one can estimate the decay rate for the solutions of system (1), using some approaches for neutral type systems; see, e. g., Hale and Verduyn-Lunel (1993); Kharitonov (2005); Kharitonov et al. (2005); Michiels and Niculescu (2014); Chashnikov and Egorov (2015). Obviously, in this case, it is important to choose additional spectrum to be located sufficiently far to the left of the imaginary axis of the complex plane. In particular, if $h = 1$, $M_1 = -I$, $M_2 = -e^{-1}I$, the spectral abscissa for the additional dynamics is equal to 1.

5.2 The $\mathcal{H}_2$ Norm Characterizing

Here, for the sake of simplicity, we consider a controllable system, satisfying Assumption 1, but the same ideas can be also applied to the analysis of strangeness-free systems (i. e., satisfying Assumption 4). Let

$$\begin{aligned} \frac{d}{dt}\big(Ex(t)\big) &= A_0x(t) + A_1x(t-h) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{10}$$

where $u(t) \in \mathbf{R}^p$ is the input, $y(t) \in \mathbf{R}^v$ is the output, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{v\times n}$, be exponentially stable. Construct the corresponding neutral type system

$$\begin{aligned} \frac{d}{dt}\Big((E + XPA_0)z(t) + XPA_1z(t-h)\Big) & \\ = A_0z(t) + A_1z(t-h) + Bu(t)&, \\ y(t) = Cz(t)&, \end{aligned} \tag{11}$$

where matrices X and P computed like in Subsection 3.1 for a Hurwitz matrix M. Also introduce the auxiliary system

$$\begin{aligned} \frac{d}{dt}\Big((E + XPA_0)z(t) + XPA_1z(t-h)\Big) & \\ = A_0z(t) + A_1z(t-h) + XPBu(t)&, \\ y(t) = Cz(t)&. \end{aligned} \tag{12}$$

Proposition 13. If the transfer function of system (12) is equal to zero, the $\mathcal{H}_2$ norm

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left(G^*(i\omega)G(i\omega)\right)\, d\omega} \tag{13}$$

of the transfer function of system (10) is finite and is equal to the one of system (11).

Proof. To prove the result we just need to explicitly express the transfer functions

$$G(s) = CH^{-1}(s)B,$$
$$G_1(s) = CH_1^{-1}(s)B,$$
$$G_2(s) = CH_1^{-1}(s)XPB$$

for systems (10), (11), (12), respectively. Using equality (6), one can obtain

$$G(s) = G_1(s) - sG_2(s). \tag{14}$$

Now the desired result follows immediately from the fact that the $\mathcal{H}_2$ norm of the transfer function of any neutral type system is finite.

Note that the most common assumption on matrix B that the "non-differential" part of the system is uncontrollable is included in the proposition.

Corollary 14. The transfer function of system (12) is equal to zero, if the "non-differential" part of the system is not affected by the input, i. e.,

$$PB = 0. \tag{15}$$

Note that the condition of Proposition 13 is necessary and sufficient for the equivalence between transfer matrices of systems (10) and (11), but this is not necessary for the finiteness of the $\mathcal{H}_2$ norm for system (10).

Proposition 15. The $\mathcal{H}_2$ norm for system (10) is finite if and only if the $\mathcal{H}_2$ norm of product $sG_2(s)$ is finite.

In particular, this is the case, if

$$C = \widetilde{C}E, \tag{16}$$

i. e., C^T belongs to the image of matrix E^T.

Proof. The first part of the proposition is obvious, as in (14) G_1 is always of finite $\mathcal{H}_2$ norm.

If $C = \widetilde{C}E$, we can express

$$\begin{aligned} sG_2(s) =& s\widetilde{C}EH_1^{-1}(s)XPB \\ =& \widetilde{C}(I - sXP)^{-1}XPB \\ &+ \widetilde{C}(A_0 + A_1e^{-sh})H_1^{-1}(s)XPB \\ =& \widetilde{C}X(I - sM^{-1})^{-1}PB \\ &+ \widetilde{C}(A_0 + A_1e^{-sh})H_1^{-1}(s)XPB. \end{aligned}$$

The first summand is of a finite $\mathcal{H}_2$ norm, because it can be considered as the transfer matrix of an exponentially stable ordinary differential system whose $\mathcal{H}_2$ norm is always finite. The last summand is also of a finite $\mathcal{H}_2$ norm, as this is the transfer matrix of an exponentially stable neutral type system with the delayed output.

If the condition of Proposition 13 holds, one can apply the following formula from Jarlebring et al. (2011) to compute the $\mathcal{H}_2$ norm for system (10):

$$\|G\|_{\mathcal{H}_2} = \sqrt{\mathrm{Tr}\left(B^TU(0)B\right)}, \tag{17}$$

where $U(0)$ is the delay Lyapunov matrix, associated with $W = C^TC$, at zero point; see, Kharitonov (2013). Note that the $\mathcal{H}_2$ norm for system (12) also can be computed by similar formula with the same $U(0)$.

6. ILLUSTRATIVE EXAMPLES

Example 16. Consider the following system from Logemann (1998):

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \dot{x}(t) = -x(t) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x(t-h) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t).$$

Let $y(t) = x(t)$. Rank of the matrix E is equal to 1. Following the algorithm of Section 3, compute

$$P = (1\ 1),$$

choose $M = -2$, and find

$$X = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The rank of the matrix

$$E + XPA_0 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

is equal to 2. By Theorem 5, the system satisfies Assumption 1, and can be reduced to system (11):

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \dot{x}(t) = -x(t) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x(t-h) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t),$$

which is of retarded type. The systems is exponentially stable, if $h < 1.209$. Take, for instance, $h = 1$. By Corollary 14 and formula (17), the $\mathcal{H}_2$ norm for the system

$$\|G\|_{\mathcal{H}_2} \approx 1.783.$$

Example 17. Consider exponentially stable system (10) with the following matrices:

$$E = \begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \\ -6 & 3 & 9 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & -3 \\ -3 & -3 & 3 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 4 & 3 \\ 5 & -4 & -8 \end{pmatrix},$$

$$B = \begin{pmatrix} 5 \\ -5 \\ 2 \end{pmatrix}, \quad C = (4 \; 6 \; -3).$$

The rank of the matrix E is equal to 1. Compute

$$P = \begin{pmatrix} 3 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

choose $M = -I$, and find

$$X = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Rank of the matrix

$$E + XPA_0 = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 4 & 3 \\ 5 & -4 & -8 \end{pmatrix}$$

is equal to 3. Therefore, by Theorem 5, the system satisfies Assumption 1.

Now we can construct neutral type systems (11) and (12). Note that neither condition (15) nor condition (16) hold, but the $\mathcal{H}_2$ norm for system (10) is finite and is equal to the $\mathcal{H}_2$ norm for system (11), as the transfer function of (12) is equal to zero. By formula (17), we find

$$\|G\|_{\mathcal{H}_2} \approx 1.305.$$

7. CONCLUSION

A neutral type system with a prescribed additional dynamics for the nominal strangeness-free system of DDAEs is constructed. As the spectra of these systems are simply related, one can analyze stability of the delay differential-algebraic system via the analysis of the corresponding neutral type system. Also characterizing of $\mathcal{H}_2$ norm of transfer matrix and estimation of the decay rate for the solutions can be achieved by the presented approach.

REFERENCES

Andronov, A.A. and Maier, A.G. (1946). The simplest linear systems with retardation. *Automatika i Telemekhanika*, 7(2-3), 95–106. In Russian. Submitted in 1939.

Chashnikov, M.V. and Egorov, A.V. (2015). Exponential estimates for linear differential-difference neutral type systems. In *Proceedings of the 2015 International Conference on "Stability and Control Processes" in Memory of V. I. Zubov*, 281–284. Saint-Petersburg, Russia.

Du, N.H., Linh, V.H., Mehrmann, V., and Thuan, D.D. (2013). Stability and robust stability of linear time-invariant delay differential-algebraic equations. *SIAM J. on Matrix Analysis and Applications*, 34(4), 1631–1654.

Fridman, E. (2002). Stability of linear descriptor systems with delay: A Lyapunov-based approach. *J. of Mathematical Analysis and Applications*, 273(1), 24–44.

Gomez, M.A., Egorov, A.V., and Mondié, S. (2016). Necessary stability conditions for neutral type systems with a single delay. *IEEE Trans. on Automatic Control, DOI: 10.1109/TAC.2016.2625738.*

Gu, K. and Liu, Y. (2009). Lyapunov–Krasovskii functional for uniform stability of coupled differential-functional equations. *Automatica*, 45(3), 798–804.

Hale, J.K. and Verduyn-Lunel, S.M. (1993). *Introduction to Functional Differential Equations.* Springer Science + Business Media, New York.

Han, Q.L., Yu, X., and Gu, K. (2004). On computing the maximum time-delay bound for stability of linear neutral systems. *IEEE Trans. on Automatic Control*, 49(12), 2281–2285.

Jarlebring, E., Vanbiervliet, J., and Michiels, W. (2011). Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay Lyapunov equation. *IEEE Trans. on Automatic Control*, 56(4), 814–825.

Kharitonov, V., Mondié, S., and Collado, J. (2005). Exponential estimates for neutral time-delay systems: An LMI approach. *IEEE Trans. on Automatic Control*, 50(5), 666–670.

Kharitonov, V.L. (2005). Lyapunov functionals and Lyapunov matrices for neutral type time delay systems: a single delay case. *Int. J. of Control*, 78(11), 783–800.

Kharitonov, V.L. (2013). *Time-delay systems. Lyapunov functionals and matrices.* Birkhäuser, Basel.

Kunkel, P. and Mehrmann, V. (2006). *Differential–Algebraic Equations. Analysis and Numerical Solution.* European Mathematical Society, Zürich.

Logemann, H. (1998). Destabilizing effects of small time delays on feedback-controlled descriptor systems. *Linear Algebra and its Applications*, 272(1-3), 131–153.

Michiels, W. and Niculescu, S.I. (2014). *Stability, control, and computation for time-delay systems. An eigenvalue-based approach.* SIAM, Philadelphia.

Michiels, W. and Vyhlídal, T. (2005). An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type. *Automatica*, 41(6), 991–998.

Pepe, P., Karafyllis, I., and Jiang, Z.P. (2008). On the Lyapunov–Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equations. *Automatica*, 44(9), 2266–2273.

Răsvan, V. (1995). Dynamical systems with lossless propagation and neutral functional differential equations. In *Proceedings of the 13th International Symposium on the Mathematical Theory of Networks and Systems*, 527–531. Padova, Italy.